

Temporal sequence storage capacity of time-summing neural networks

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1992 J. Phys. A: Math. Gen. 25 833

(<http://iopscience.iop.org/0305-4470/25/4/020>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.59

The article was downloaded on 01/06/2010 at 17:53

Please note that [terms and conditions apply](#).

Temporal sequence storage capacity of time-summing neural networks

P C Bressloff[†] and J G Taylor[‡]

[†] Hirst Research Centre, GEC-Marconi Ltd, East Lane, Wembley, Middlesex HA9 7PP, UK

[‡] Department of Mathematics, King's College, London University, Strand, London, WC2 R 2LS, UK

Received 15 April 1991

Abstract. The maximum capacity for storage of temporal sequences in a time-summing neural network is analysed within a statistical-mechanical framework. Each neuron of such a network maintains an activity trace consisting of a decaying sum of all previous inputs to that neuron. The maximum storage capacity is expressed as a function of the temporal correlations of the local fields of the neurons and evaluated using a perturbation expansion in powers of the decay rate.

1. Introduction

The enormous upsurge of research into artificial neural networks has been based, by and large, on a very simple, discrete-time model of real neurons. In particular, the fundamental unit of a binary Hopfield network [1] is a two-state element which is either on or off depending upon whether or not the total input to the unit at the previous time-step exceeds some threshold. However, there is growing interest in studying discrete-time networks in which the individual neurons have some memory of previous states of the network which extends beyond a single time-step [2–8]. The simplest way to achieve this is to take the neurons to be capacitive so that the total input to a neuron is the sum of the present input together with a fractional part of the previous input. This results in the neuron maintaining an activity trace consisting of a decaying sum of all previous inputs to that neuron. Such a model is a discrete-time version [6] of the leaky-integrator model used in analogue Hopfield networks [1].

There are a number of reasons why these so-called time-summing neurons are of interest. Firstly, recent neurophysiological evidence [9] suggests that the persistence of activity on the cell membrane of certain neurons in the cortex occurs over hundreds of milliseconds, which is considerably higher than had previously been thought. (A single time-step is of the order of one millisecond.) Secondly, time-summing networks can display a wide range of dynamical behaviour including frequency-locking and chaos [7, 8] and provide a discrete-time alternative to the coupled-oscillator systems based upon analogue Hopfield networks [10].

A third reason for studying time-summing networks, which is the motivation for this paper, is that in the case of synchronous (parallel) dynamics, such networks have certain advantages over standard models in the processing of temporal sequences. For example, the deterministic dynamics of a binary Hopfield network [1] with parallel update is characterized by a unique, single-step, state transition matrix so that it can

only store simple sequences in which a succession of states is unambiguously defined. On the other hand, the ambiguities arising from repeated patterns can be resolved by time-summing neurons, provided incoming activity is held over a long enough period [2, 5]. Another advantage of the latter is that simple on-line learning algorithms have been developed which deal with temporal sequences directly in the time-domain [3-5] so avoiding the need for mapping the temporal patterns into spatial ones of much higher dimension.

We note that an alternative approach to temporal sequence storage has been developed for Hopfield networks with asynchronous (sequential) dynamics [11-13]. This involves introducing additional couplings between the neurons which have signal transmission delays associated with them. If such an approach is applied to the case of parallel dynamics [14] then one obtains a model which is similar in form to that of a time-summing network. However, the time-period over which previous inputs to a neuron are summed is restricted to the number of delay lines between neuron pairs. On the other hand, the neurons of a time-summing network maintain an activity trace which is of arbitrary length and which does not require the introduction of additional connections.

In this paper we analyse the capacity for temporal sequence storage of a Hopfield-like network with parallel dynamics which has been modified by taking the elements of the network to be time-summing neurons. We show how Gardner's [15] statistical-mechanical framework for the space of connection weights between the neurons can be extended to the time-summing case. We derive an explicit expression for the maximum storage capacity which is a function of the temporal correlations of the local fields of the neurons. We also construct a perturbation expansion of this expression in terms of the rate of decay of inputs to the neurons and use this to show how the presence of extended time-summation leads to a reduction, relative to standard Hopfield networks, in the maximum capacity for storage of uncorrelated simple temporal sequences in a fully-connected network. The size of the reduction increases with κ , where κ is the stability parameter determining the size of the basins of attraction [15]. Moreover, for $\kappa = 0$ the critical storage capacity is independent of d and coincides with the result for a standard Hopfield network. This indicates that the mechanism by which extended time-summation affects the critical storage capacity is through an effective rescaling of the stability parameter κ . The reduction for $\kappa > 0$ is a consequence of the nontrivial temporal correlations between the local fields. This follows from the result that if the network is sparsely connected such correlations become unimportant and one finds an increase in storage capacity for $\kappa > 0$.

2. Space of interactions

We proceed by considering a network of N time-summing binary-threshold neurons [5, 6]. Denote the state of the network at the discrete time t by the N -component 'spin' vector $S(t) = (S_1(t), \dots, S_N(t))$ where $S_i(t) = \pm 1$ is the output of neuron i . Assume that the neurons are connected by synaptic weights J_{ij} . The dynamics of such a network in the case of zero noise and vanishing thresholds is given by

$$S_i(t) = \text{sgn}(h_i(t)), \quad h_i(t) = \frac{1}{\sqrt{N}} \sum_{j \neq i} J_{ij} \sum_{r=1}^t S_j(r-1) d^{t-r} \quad (1)$$

where $h_i(t)$ is the local field of neuron i and d , $d < 1$, determines how rapidly the

inputs to each neuron decay with time. In the limit $d \rightarrow 0$ equation (1) reduces to the usual dynamical rule for Hopfield networks [1], in which the output of the neuron at time t only depends on the state of the network at the previous time-step. Note that the weights J_{ij} can be taken to be time-dependent in which case $J_{ij}d^{t-r}$ is replaced by $J_{ij}(t-r)$ [16]. Equation (1) may be rewritten as an iterative equation for the local fields which for constant weights is given by

$$h_i(t) = dh_i(t-1) + N^{-1/2} \sum_{j \neq i} J_{ji} \operatorname{sgn}(h_j(t-1)). \tag{2}$$

The dynamics of equation (2), and certain nonlinear generalizations thereof [6], have been studied elsewhere [7, 8] and can be shown to exhibit complex oscillatory and chaotic behaviour. We see from (2) that the mechanism by which a neuron develops an activity trace of previous inputs is by means of a slowly decaying local field or membrane potential. This should be contrasted with the time-delay approach [14] which introduces additional delay-lines between the neurons. The dynamics of the latter type of model is described by equation (1) but with the time-summation restricted to $r \leq t_{\max}$, where t_{\max} is the maximum number of delay-lines between neuron pairs. Such a truncated expression cannot be rewritten in the simple iterative form of equation (2).

In this paper we are concerned with the problem of choosing a set of weights J_{ij} such that $p = \alpha N$ prescribed temporal sequences $\{\xi_i^\mu(t) = \pm 1; t = 0, 1, \dots, T\}$, $\mu = 1, \dots, p$, $i = 1, \dots, N$, of length $T+1$ are stored, where T is arbitrary. We assume that on recall each sequence is seeded by the first element of that sequence. No periodicity conditions are imposed on the class of such sequences. Introducing a stability parameter κ , taken to be a positive constant, we require that a solution set of weights $\{J_{ij}\}$ satisfies the αNT conditions

$$\xi_i^\mu(t) \left[\sum_{j \neq i} J_{ij} \sum_{r=1}^t \xi_j^\mu(r-1) d^{t-r} \right] > \kappa \quad 1 \leq t \leq T \tag{3}$$

which are a natural extension of the fixed point conditions for standard Hopfield networks. Following the statistical approach of Gardner [15], we shall calculate the typical fractional volume of the space of solutions $\{J_{ij}\}$ to equation (3) given that the weights are normalized so that $\sum_{j \neq i} J_{ij}^2 = N$. The critical value α_c of α above which the volume vanishes then determines the maximum storage capacity of the network.

The fractional volume V for a given set of patterns is given by $V = \prod_{i=1}^N V_i$ where

$$V_i = \frac{\int \prod_{j \neq i} dJ_{ij} \prod_{\mu,r} \theta(\xi_i^\mu(r) \sum_{j \neq i} J_{ij} \sum_{s=1}^r \xi_j^\mu(s-1) d^{r-s} - \kappa) \delta(\sum_{j \neq i} J_{ij}^2 - N)}{\int \prod_{j \neq i} dJ_{ij} \delta(\sum_{j \neq i} J_{ij}^2 - N)} \tag{4}$$

In the thermodynamic limit the typical fractional volume can be calculated from the average of $\log V_i$ over the quenched distributions of the pattern sequences $\{\xi_j(t); t = 0, 1, \dots, T\}$, $j = 1, \dots, N$ for fixed i . This may be achieved using the replica method [15], $\langle \log V_i \rangle_\xi = \lim_{n \rightarrow 0} [\langle V_i^n \rangle_\xi - 1] / n$. If the analytic continuation in n from positive integer values to zero is valid then V_i^n may be obtained from (4) by adding a replica index to each of the weights J_{ij} and taking a product over n replicas. Using the usual integral representation of the theta-functions one can average over the distributions of pattern sequences explicitly. For simplicity we take the $\xi_i(t)$, $i = 1, \dots, N$, $t = 0, \dots, T$, to be independent random variables such that for each pattern μ , $\xi_i^\mu(t) = \pm 1$ with equal probability. Assuming symmetry between the replicas we find in the large- N limit, as

n tends to zero, that

$$\langle V_i^n \rangle_\xi = \exp \left(Nn \min_q G(q) + O \left(\frac{1}{N} \right) \right) \tag{5}$$

where

$$G(q) = \frac{q}{2(1-q)} + \frac{1}{2} \log(1-q) + \alpha \left\langle \int \mathcal{D}t \log \left[\int \mathcal{D}y \prod_{r=1}^T \theta \left(y_r - \frac{\sqrt{q} t_r + \kappa}{\sqrt{1-q}} \right) \right] \right\rangle_\xi \tag{6}$$

and the measures $\mathcal{D}t, \mathcal{D}y$ are of the form

$$\mathcal{D}t = \prod_{r=1}^T \frac{dt_r}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} \sum_{r,r'} \xi_i(r) t_r \mathbf{D}_{rr'}^{-1} t_{r'} \xi_i(r') \right) \tag{7}$$

and similarly for $\mathcal{D}y$. (See appendix for a detailed derivation of equation (5).)

The $T \times T$ matrix \mathbf{D} of equation (7) is symmetric, invertible, has positive definite eigenvalues and unit determinant. It may be interpreted as the temporal correlation matrix of the local field of neuron i when $S(r) = \xi(r)$, $r = 0, \dots, T$, i.e. $\mathbf{D}_{rr'} = \langle h_i(r) h_i(r') \rangle_\xi$. In component form,

$$\mathbf{D}_{rr'} = d^{r+r'} \sum_{s=1}^{\min(r,r')} d^{-2s} \quad \mathbf{D}_{rr'}^{-1} = [1 + d^2(1 - \delta_{r,T})] \delta_{r,r'} - d[\delta_{r,r'+1} + \delta_{r+1,r'}]. \tag{8}$$

The variable q is the Edwards–Anderson order parameter [15] $q = \sum_{j \neq i} \langle (J_{ij})^2 \rangle_T \rangle_\xi$ where $\langle \dots \rangle_T$ means the average over all parameters $\{J_{ij}, j \neq i\}$ which are solutions to equation (3) and is determined from the saddle point equation $\partial G / \partial q = 0$. For $q \rightarrow 1$ this leads to the maximum value α_c of α , i.e. the critical capacity α_c is obtained when the fractional volume shrinks to zero, and this happens when there is only one solution as indicated by $q \rightarrow 1$. Note that an alternative expression for $G(q)$ was previously derived in [5]. However, equation (6) makes the calculation of the critical storage capacity much more tractable and, in particular, allows one to evaluate the contribution from off-diagonal terms $\mathbf{D}_{rr'}$, $r \neq r'$, which was not possible in the earlier work. Indeed, it will turn out that such contributions are important.

3. Storage capacity

We now discuss the analysis of the critical storage capacity $\alpha_c^{(d)}$ for $d \geq 0$. Let $x_r(t, q) = (\sqrt{q} t_r + \kappa) / \sqrt{1-q}$, set $x = (x_1, \dots, x_T)$, and denote the argument of the logarithm in equation (6) by $H(x(t, q))$. The saddle point equation then becomes

$$\frac{q}{2(1-q)^2} = -\alpha^{(d)} \left\langle \int_{-\infty}^{\infty} \mathcal{D}x H(x)^{-1} \sum_{s=1}^T \frac{\partial H}{\partial x_s} \frac{\partial x_s}{\partial q} \right\rangle_\xi. \tag{9}$$

To establish the behaviour of the integrand in equation (9) as $q \rightarrow 1$, we asymptotically expand $H(x)$ with respect to x_r , $r = 1, \dots, T$, for $x_r \rightarrow \pm\infty$. Suppose that $x_r \rightarrow +\infty$ for all $r \in I$ and $x_r \rightarrow -\infty$ for all $r \notin I$ where I is some non-empty subset of $\{1, \dots, T\}$. To leading order, the integration range of the variables y_r , $r \notin I$, in equation (6) may be taken to be $(-\infty, \infty)$ and hence these variables can be integrated explicitly by repeatedly completing the square. (Such integrals are well-behaved since the eigenvalues of the matrix \mathbf{D} are positive definite). If the resulting expression for $H(x)$ is then asymptotically expanded with respect to x_r , $r \in I$ we obtain the leading order approximation

$$H(x) \approx \mathcal{A}^{(1)} \left(\prod_{r \in I} \left[\sum_{r' \in I} \mathbf{D}_{rr'}^{(1)} x_r \xi(r) \xi(r') \right] \right)^{-1} \exp \left(-\frac{1}{2} \sum_{r,r' \in I} \xi(r) x_r \mathbf{D}_{rr'}^{(1)} x_{r'} \xi(r') \right) \tag{10}$$

where the $T \times T$ matrix $\mathbf{D}^{(1)}$ may be defined iteratively as follows: if the complement of the set I is $\{j_1, \dots, j_p : p < T\} \subset \{1, \dots, T\}$ then $\mathbf{D}_{rr'}^{(1)} = \hat{\mathbf{D}}_{rr'}^{(j_1, \dots, j_p)}$ where

$$\hat{\mathbf{D}}_{rr'}^{(j_1, \dots, j_p)} = \hat{\mathbf{D}}_{rr'}^{(j_2, \dots, j_p)} - \frac{\hat{\mathbf{D}}_{rj_1}^{(j_2, \dots, j_p)} \hat{\mathbf{D}}_{r'j_1}^{(j_2, \dots, j_p)}}{\hat{\mathbf{D}}_{j_1 j_1}^{(j_2, \dots, j_p)}}, \dots, \hat{\mathbf{D}}_{rr'}^{(j_p)} = \mathbf{D}_{rr'}^{-1} - \frac{\mathbf{D}_{rj_p}^{-1} \mathbf{D}_{r'j_p}^{-1}}{\mathbf{D}_{j_p j_p}^{-1}}. \tag{11}$$

The factor $\mathcal{A}^{(1)}$ in equation (10) is a normalization factor involving determinants, and will be cancelled by an identical factor arising from the asymptotic expansion of $\partial H / \partial x_s$. Note that the formula for $\hat{\mathbf{D}}_{rr'}^{(j_1, \dots, j_p)}$ in equation (11) is invariant under all permutations of the indices j_1, \dots, j_p . Similar expansions may be performed for $\partial H / \partial x_s$, $s = 1, \dots, T$, from which it can be deduced that, for a given I , the contribution of the term involving $H^{-1}(x) \partial H / \partial x_s$ to the t -integration in equation (9) vanishes as $q \rightarrow 1$ unless $s \in I$. If this condition is satisfied then $H^{-1}(x) \partial H / \partial x_s \approx \sum_{r \in I} \mathbf{D}_{sr}^{(1)} x_r \xi(r) \xi(s)$. Finally, summing over all possible choices for I , we find that in the limit $q \rightarrow 1$ equation (9) reduces to

$$1 = \alpha_c^{(d)}(\kappa) \left\langle \mathcal{D}t \sum_I \left\{ \Theta_I(t) \sum_{r, r' \in I} (t_r + \kappa) \mathbf{D}_{rr'}^{(1)} (t_r + \kappa) \xi(r) \xi(r') \right\} \right\rangle_\xi \tag{12}$$

where $\Theta_I(t) = \prod_{r \in I} \theta(t_r + \kappa) \prod_{r' \notin I} \theta(-t_{r'} - \kappa)$ and $\mathcal{D}t$ is the integration measure of equation (7).

The first point to note about equation (12) is that when $\kappa = 0$ the t -integrations can be performed explicitly leading to the result that $\alpha_c^{(d)}(0) = 2/T$ for all d , $0 \leq d \leq 1$. To show this we use the fact that for $\kappa = 0$ the integration ranges of all the t_r 's can be extended to $(-\infty, \infty)$. We may then integrate over each $t_r \notin I$ by completing the square in an identical fashion to the derivation of equation (10) giving

$$\begin{aligned} 1 &= \alpha_c^{(d)}(0) \left\langle \sum_I \frac{1}{2^T} \mathcal{A}^{(1)} \int_{-\infty}^{\infty} \prod_{r \in I} dt_r \left[\sum_{s, s' \in I} \xi(s) t_s \mathbf{D}_{ss'}^{(1)} t_{s'} \xi(s') \right] \right. \\ &\quad \times \exp \left[-\frac{1}{2} \sum_{r, r' \in I} \xi(r) t_r \mathbf{D}_{rr'}^{(1)} t_{r'} \xi(r') \right] \left. \right\rangle_\xi \\ &= \alpha_c^{(d)}(0) \sum_I \frac{1}{2^T} C(I) \mathcal{A}^{(1)} [\det \mathbf{D}^{(1)}]^{-1/2} \end{aligned}$$

where $C(I)$ is the number of elements of the set I . The result then follows since $\mathcal{A}^{(1)} [\det \mathbf{D}^{(1)}]^{-1/2} = [\det \mathbf{D}]^{-1/2} = 1$ and $\sum_I C(I) = T 2^{T-1}$. A second feature of equation (12) is that in the limit $d \rightarrow 0$ the matrices \mathbf{D} and $\mathbf{D}^{(1)}$ reduce to the $T \times T$ unit matrix and (12) becomes

$$\alpha_c^{(0)}(\kappa) = \frac{1}{T} \left[\int_{-\kappa}^{\infty} Dt (t + \kappa)^2 \right]^{-1} \tag{13}$$

which is identical to the result of Gardner [15] for the storage of static patterns in a Hopfield network, except for the modification that α is multiplied by an extra factor of T . This is to be expected since the network is effectively storing αNT uncorrelated sequences of step length two when $d = 0$.

For non-zero values of the decay rate d we may use the fact that $d < 1$ to expand the integrand of equation (12) in powers of d and perform the average over the patterns $\xi(r)$ using $\langle \xi(r) \xi(r') \rangle = \delta_{rr'}$. To second order in d , the matrix $\mathbf{D}^{(1)}$ simplifies considerably to give

$$\mathbf{D}_{rr'}^{(1)} = \mathbf{D}_{rr'}^{-1} - d^2 \sum_{j \notin I} [\delta_{r, j+1} + \delta_{r+1, j}] [\delta_{r', j+1} + \delta_{r'+1, j}] + O(d^3). \tag{14}$$

After some algebra we obtain

$$\frac{\alpha_c^{(0)}(\kappa)}{\alpha_c^{(d)}(\kappa)} = 1 + \frac{d^2(T-1)}{2T} \left\{ \left[1 - 4 \int_{-\infty}^{-\kappa} Dt \right] + T\alpha_c^{(0)}(\kappa) \right. \\ \left. \times \left[\int_{-\kappa}^{\infty} Dt(t+\kappa)^2 t^2 - 4 \left(\int_{-\kappa}^{\infty} Dt(t+\kappa)t \right)^2 \right] \right\} + O(d^4). \quad (15)$$

The expression in curly brackets on the right-hand side of equation (15), $F(\kappa)$ say, is a monotonically increasing function of the stability parameter κ , increasing from the value 0 at $\kappa=0$ to the maximum value 2 as $\kappa \rightarrow \infty$, (see figure 1). Hence, to second order in the decay rate d , $\alpha_c^{(d)}(\kappa) \leq \alpha_c^{(0)}(\kappa)$ for all κ , with the equality only holding at $\kappa=0$ when $\alpha_c^{(0)}(0) = 2/T$. In other words, for small decay rates and non-zero values of the stability parameter κ , the presence of extended time-summation leads to a reduction, relative to standard Hopfield networks, in the maximum capacity for storage of uncorrelated temporal sequences. A numerical evaluation of equation (12) for $T=2$ shows that this reduction occurs for all values of d , $0 \leq d \leq 1$, (see figure 2). Moreover, the size of the reduction increases monotonically with κ from zero at $\kappa=0$. A physical interpretation of these results, which is expected to be true for all $T > 1$ and $d \leq 1$, is that the persistence of network activity over a number of time-steps leads to interference between different patterns of a sequence, as expressed by the off-diagonal elements of the temporal correlation matrix \mathbf{D} , which reduces the storage capacity for $\kappa > 0$. The fact that the storage capacity is independent of d for $\kappa=0$ suggests that this reduction is due to an effective rescaling of the stability parameter.

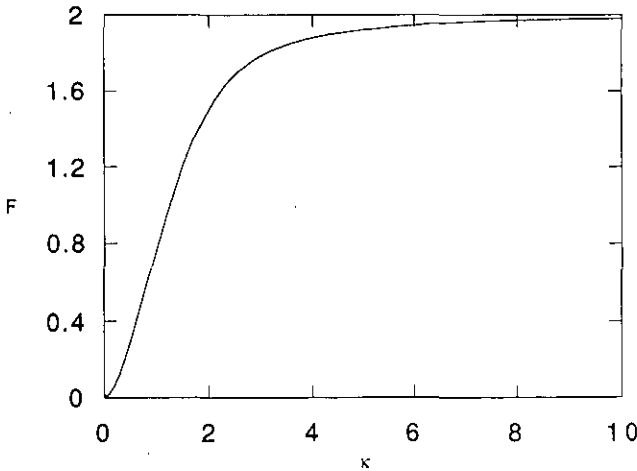


Figure 1. The $O(d^2)$ contribution to $\alpha_c^{(0)}(\kappa)/\alpha_c^{(d)}(\kappa)$ as a function of the stability parameter κ , in units of $d^2(T-1)/T$.

To confirm the above picture, it is interesting to note that if the off-diagonal contributions can be neglected then the maximum storage capacity is enhanced, rather than reduced, due to the fact that the diagonal elements of \mathbf{D} lead to a lower effective stability parameter κ , i.e. κ is rescaled as $\kappa(\mathbf{D}_{rr})^{-1/2}$ (see appendix). One situation in which the off-diagonal contributions are negligible [5] is for a sparsely-connected network with definite symmetry [17]. In such a model each neuron is connected, on

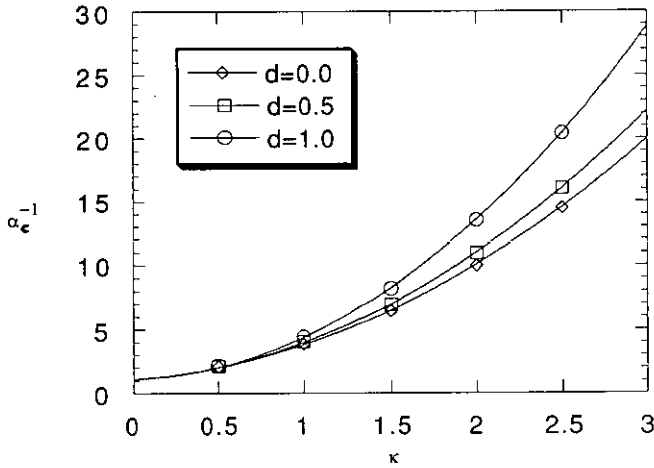


Figure 2. The inverse maximum storage capacity α_c^{-1} as a function of the stability parameter κ in the case of sequences of length $T = 2$, showing the dependence on the decay rate d .

the average, to C other neurons, where C is of order $\log N$, and the existence of a connection from i to j implies also the existence of a connection from j to i according to some symmetry constraint. The normalization condition for the weights is modified to $\sum_{j \neq i} (J_{ij})^2 = C$ and the summation over j restricted to the C sites connected to i . In contrast to the fully-connected network analysed in section 2, one cannot decouple the contributions to the typical fractional volume from the various sites i since the symmetry constraint implies that the weights J_{ij} and J_{ji} are correlated. It is necessary, therefore, to evaluate the full expression $\langle V^n \rangle$ where $V = \prod_i V_i$ and V_i is given by equation (4), (once the normalization and symmetry conditions have been included). The average over the uncorrelated patterns $\xi_i, i = 1, \dots, N$, is performed by introducing integral representation of the theta-functions in the usual way (cf equation (A2)) and, for each μ , leads to an expression of the form

$$\prod_{i,\alpha} \int_{-\kappa}^{\infty} \frac{dy_{i,\mu,r}^\alpha}{2\pi} \int_{-\infty}^{\infty} dx_{i,\mu,r}^\alpha e^{-ix_{i,\mu,r}^\alpha y_{i,\mu,r}^\alpha} \times \left\langle \exp \left[i \sum_{i,\alpha,r} x_{i,\mu,r}^\alpha \frac{1}{\sqrt{C}} \xi_i^\mu(r) \sum_{j \neq i} J_{ij}^\alpha \sum_{s=1}^r \xi_j^\mu(s-1) d^{r-s} \right] \right\rangle_\xi.$$

The expression in angular brackets may be evaluated using a cumulant expansion [17] and to leading order in $1/C$, reduces to [5]

$$\exp \left[-\frac{1}{2} \left\langle \left(\sum_{i,\alpha,r} x_{i,\mu,r}^\alpha \frac{1}{\sqrt{C}} \xi_i^\mu(r) \sum_{j \neq i} J_{ij}^\alpha \sum_{s=1}^r \xi_j^\mu(s-1) d^{r-s} \right)^2 \right\rangle_\xi \right] = \exp \left[-\frac{1}{2C} \sum_{i,\alpha,\beta} \sum_r x_{i,\mu,r}^\alpha x_{i,\mu,r}^\beta \mathbf{D}_{rr} \left(\sum_{j \neq i} J_{ij}^\alpha J_{ij}^\beta \right) \right]. \tag{16}$$

Hence, in the large- N limit the contribution from off-diagonal elements of \mathbf{D} to the typical fractional volume vanishes, (cf equation (A3)). Using a change of variables it can be shown that the diagonal elements \mathbf{D}_{rr} simply lead to a rescaling of $\kappa, \kappa \rightarrow \kappa(\mathbf{D}_{rr})^{-1/2}$ such that the storage capacity increases monotonically with d . A more detailed analysis is presented elsewhere [5].

4. Discussion

In this paper we have evaluated the maximum capacity for storage of uncorrelated temporal sequences in a fully-connected time-summing network. We have shown that, for $\kappa > 0$, the presence of non-zero temporal correlations of the neurons' local fields leads to a reduction in storage capacity compared to a standard Hopfield network. We have also contrasted this with the case of a diluted network for which an increase, rather than a decrease, in storage capacity is found. There are a number of interesting aspects of this work which need to be developed further. Firstly, one of the advantages of time-summing networks is that they allow the storage of complex sequences. However, the above analysis was concerned with simple sequences alone, since in the thermodynamic limit the probability that a finite sequence of uncorrelated patterns contains a repeated pattern approaches zero. Secondly, it might be possible to evaluate equation (12) non-perturbatively by studying more closely the matrices \mathbf{D} and $\mathbf{D}^{(1)}$. For example, in the limit $d \rightarrow 1$ the former reduces to a well-known form [18]. Finally, having extended Gardner's analysis to time-summing networks the question of generalization in such networks may be tackled [19].

Acknowledgment

We would like to thank M J Kearney of Hirst Research Centre for helpful comments during the completion of this work.

Appendix

In this appendix we present the detailed derivation of the expectation $\langle V^n \rangle$, equation (5). Taking product over n replicas, $\alpha = 1, \dots, n$, gives

$$\langle V^n \rangle_\xi = \left\langle \prod_{\alpha=1}^n \int \prod_{j \neq i} dJ_{ij}^\alpha \prod_{\mu,r} \theta \left(\frac{1}{\sqrt{N}} \xi_i^\mu(r) \sum_{j \neq i} J_{ij}^\alpha \sum_{s=1}^r \xi_j^\mu(s-1) d^{r-s} - \kappa \right) \times \delta \left(\sum_{j \neq i} (J_{ij}^\alpha)^2 - N \right) \right\rangle_\xi \left[\prod_{\alpha=1}^n \int \prod_{j \neq i} dJ_{ij}^\alpha \delta \left(\sum_{j \neq i} (J_{ij}^\alpha)^2 - N \right) \right]^{-1} \tag{A1}$$

where J_{ij}^α is the realization of J_{ij} for replica α . To evaluate (A1) we use the integral representations of the theta-functions for each sequence μ and replica α ,

$$\theta \left(\frac{1}{\sqrt{N}} \xi_i^\mu(r) \sum_{j \neq i} J_{ij}^\alpha \sum_{s=1}^r \xi_j^\mu(s-1) d^{r-s} - \kappa \right) = \int_{-\kappa}^\infty \frac{dy_{\mu,r}^\alpha}{2\pi} \int_{-\infty}^\infty dx_{\mu,r}^\alpha \times \exp \left[ix_{\mu,r}^\alpha \left(y_{\mu,r}^\alpha - \frac{1}{\sqrt{N}} \xi_i^\mu(r) \sum_{j \neq i} J_{ij}^\alpha \sum_{s=1}^r \xi_j^\mu(s-1) d^{r-s} \right) \right]. \tag{A2}$$

Averaging over the random patterns $\xi^\mu(r)$, $r = 1, \dots, T$ at the sites $j \neq i$ gives to leading order in $1/N$,

$$\exp \left[-\frac{1}{2} \sum_{\mu, \alpha, \beta} \left(\sum_{j \neq i} J_{ij}^\alpha J_{ij}^\beta / N \right) \sum_{r,r'} \xi_i^\mu(r) x_{\mu,r}^\alpha \mathbf{D}_{rr'} x_{\mu,r}^\beta \xi_i^\mu(r') \right] \tag{A3}$$

where \mathbf{D} is the temporal correlation matrix of equation (8).

Following [15] we define a variable $q_{\alpha\beta}$,

$$q_{\alpha\beta} = \frac{1}{N} \sum_{j \neq i} J_{ij}^\alpha J_{ij}^\beta \quad \alpha < \beta \tag{A4}$$

and introduce conjugate variables $F_{\alpha\beta}$ and E_α to implement, respectively, the conditions (A4) and $\sum_{j \neq i} (J_{ij}^\alpha)^2 = N$. Then

$$\begin{aligned} \langle V_i^n \rangle = & \int \prod_{\alpha=1}^n dE_\alpha \int \prod_{\alpha < \beta} \frac{dq_{\alpha\beta} dF_{\alpha\beta}}{2\pi/N} \left(\int \prod_{\alpha=1}^n dE_\alpha \exp[N(G_2(0, E_\alpha) + \frac{1}{2}E_\alpha)] \right)^{-1} \\ & \times \exp \left[\left(\alpha G_1(q_{\alpha\beta}) + G_2(F_{\alpha\beta}, E_\alpha) - \sum_{\alpha < \beta} F_{\alpha\beta} q_{\alpha\beta} + \sum_\alpha \frac{1}{2} E_\alpha \right) \right] \end{aligned} \tag{A5}$$

where

$$\begin{aligned} G_1(q_{\alpha\beta}) = & \log \prod_{\alpha=1}^n \prod_{r=1}^T \int_{-\kappa}^{\infty} \frac{dy_r^\alpha}{2\pi} \int_{-\infty}^{\infty} dx_r^\alpha \\ & \times \exp \left[i \sum_{\alpha,r} x_r^\alpha y_r^\alpha - \sum_{r,r'} \left(\frac{1}{2} \sum_\alpha x_r^\alpha x_{r'}^\alpha + \sum_{\alpha < \beta} q_{\alpha\beta} x_r^\alpha x_{r'}^\beta \right) \xi_i(r) \mathbf{D}_{rr'} \xi_i(r') \right] \end{aligned} \tag{A6}$$

and

$$G_2(F_{\alpha\beta}, E_\alpha) = \log \left[\prod_{\alpha=1}^n \int dJ^\alpha \exp \left(-\frac{1}{2} \sum_\alpha E_\alpha J_\alpha^2 + \sum_{\alpha < \beta} F_{\alpha\beta} J^\alpha J^\beta \right) \right]. \tag{A7}$$

Note that the only modification to the Gardner analysis of [15] occurs in the term G_1 . In the large- N limit, $\langle V^n \rangle$ is determined by the saddle point over the variables $F_{\alpha\beta}$, $q_{\alpha\beta}$, E_α of the argument of the second exponential in equation (A5). This saddle point is determined by imposing the replica-symmetry ansatz

$$\begin{aligned} q_{\alpha\beta} &= q & \alpha < \beta \\ F_{\alpha\beta} &= F & \alpha < \beta \\ E_\alpha &= E & \text{for all } \alpha. \end{aligned} \tag{A8}$$

Using the identity

$$\begin{aligned} \exp \left[-q \sum_{r,r'} \left(\xi_i(r) \mathbf{D}_{rr'} \xi_i(r') \sum_{\alpha < \beta} x_r^\alpha x_{r'}^\beta \right) \right] = & \int \prod_r \frac{dt_r}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} \sum_{r,r'} \xi_i(r) \mathbf{D}_{rr'}^{-1} \xi_i(r') t_r t_{r'} \right] \\ & \times \exp \left[i\sqrt{q} \sum_r t_r \sum_\alpha x_r^\alpha + \frac{1}{2} q \sum_{r,r'} \sum_\alpha x_r^\alpha x_{r'}^\alpha \mathbf{D}_{rr'} \xi_i(r) \xi_i(r') \right] \end{aligned} \tag{A9}$$

and performing the integrations over the x 's in (A6) we obtain equation (5) in the limit $n \rightarrow 0$.

We note that if the contribution to (A6) from off-diagonal elements of \mathbf{D} is neglected then G_1 decomposes into a sum over $r = 1, \dots, T$ terms of the form

$$\log \prod_{\alpha=1}^n \int_{-\kappa/\sqrt{\mathbf{D}_{rr}}}^{\infty} \frac{dy_r^\alpha}{2\pi} \int_{-\infty}^{\infty} dx_r^\alpha \exp \left[i \sum_\alpha x_r^\alpha y_r^\alpha - \left(\frac{1}{2} \sum_\alpha x_r^\alpha x_r^\alpha + \sum_{\alpha < \beta} q_{\alpha\beta} x_r^\alpha x_r^\beta \right) \right]$$

which is identical to the corresponding expression for G_1 in the case of a Hopfield network [15], i.e. $d = 0$, but with the stability parameter κ rescaled by $(\mathbf{D}_{rr})^{-1/2}$. Under such an approximation the analysis of the typical fractional volume leads to the result [5], $\alpha_c^{(d)}(\kappa) \approx T^{-1} \sum_r \alpha_c^{(0)}(\kappa/\sqrt{\mathbf{D}_{rr}}) > \alpha_c^{(0)}(\kappa)$ for $\kappa > 0$. However, for a fully-connected network the off-diagonal contributions are non-negligible and, as shown in this paper, $\alpha_c^{(d)}(\kappa) < \alpha_c^{(0)}(\kappa)$ for $\kappa > 0$.

References

- [1] Hopfield J J 1982 *Proc. Natl Acad. Sci. USA* **79** 2554; 1984 *Proc. Natl Acad. Sci. USA* **81** 3088
- [2] Guyon, I, Personnaz, L, Nadal J P and Dreyfus G 1988 *Phys. Rev. A* **38** 6365
- [3] Stornetta W, Hogg T and Huberman B A 1987 *Neural Information Processing Systems* ed D Z Anderson (New York: American Institute of Physics)
- [4] Mozer M C 1989 *Complex Systems* **3** 349
- [5] Reiss M and Taylor J G 1991 in preparation
Taylor J G 1987 *Int. J. Neural Syst.* **2** 47
- [6] Bressloff P C and Taylor J G 1990 *Proc. Int. Conf. on Neural Networks (Paris)* (Dordrecht: Kluwer)
- [7] Bressloff P C and Stark J 1990 *Phys. Lett.* **150A** 187
- [8] Aihara K, Takabe T and Toyoda M 1990 *Phys. Lett.* **144A** 333
- [9] Major G, Larkman A and Jack J 1990 *Proc. Physiol. Soc.* **23**
- [10] Li Z and Hopfield J J 1989 *Biol. Cybern.* **61** 379
Atiya A and Baldi P 1989 *Int. J. Neural Syst.* **1** 103
- [11] Kleinfeld D 1986 *Proc. Natl Acad. Sci. U.S.A.* **83** 9469
- [12] Sompolinsky H and Kanter I 1986 *Phys. Rev. Lett.* **57** 2861
- [13] Herz A V M, Sulzer B, Kuhn R and van Hemmen J L 1988 *Europhys. Lett.* **7** 663; 1989 *Biol. Cybern.* **60** 457
- [14] Herz A V M, Li Z and van Hemmen J L 1991 *Phys. Rev. Lett.* **66** 1370
- [15] Gardner E 1988 *J. Phys. A: Math. Gen.* **21** 257
- [16] Taylor G 1990 *Proc. Int. Conf. on Neural Networks (Paris)* (Dordrecht: Kluwer)
- [17] Gardner E, Gutfreund H and Yekutieli I 1989 *J. Phys. A: Math. Gen.* **22** 1995
- [18] Ohberge I G and Krein M G 1969 *Trans. Math. Monographs Am. Math. Soc.* **18**; 1970 **24**
- [19] Bressloff P C 1991 *Phys. Rev. A* submitted